Technical Notes

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Extension of the Blasius Force Theorem to Subsonic Speeds

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Introduction

THE Blasius theorem[1,2] is a well-known method for calculating the force on a body situated in an incompressible, inviscid two-dimensional flow. The method has numerous applications and is particularly useful in cases where there are several bodies and singularities (i.e., sources, sinks, and vortices) in the flow, since it gives the force acting separately on each body and singularity. The efficiency of the Blasius theorem is due to its quality of expressing the forces with the aid of contour integrals of analytic functions of complex variables.

The objective of the present Note is to deduce an analog of Blasius theorem for the aerodynamic forces in subsonic flow. For this purpose, we assume that an approximate velocity potential of the subsonic flow has been calculated by using the Imai-Lamla method.[3] This method is a variant specially suited for the two-dimensional flows of the Janzen-Rayleigh expansion method.[4] Then the aerodynamic forces on a body, with possible surface blowing or suction, are evaluated in terms of the velocity potential. The resulting formula expresses the aerodynamic forces with the aid of contour integrals of analytic complex functions and it can be regarded as the Blasius theorem with first-order compressibility correction for the subsonic speed regime. Finally, the application of the subsonic Blasius theorem to cases in which there are compressible vortices in the flow is discussed and an example is presented.

Summary of the Imai-Lamla Method

We shall recall the elements of the Imai-Lamla approximation method[3] that will be used in the next section to evaluate the aerodynamic forces on a body on subsonic flow.

In applying the Imai-Lamla method, the coordinates x and y are first changed into the complex variables

$$z = x + iy; \qquad \bar{z} = x - iy \tag{1}$$

Then, for a plane, steady and potential flow, the complex velocity potential and its conjugate are introduced

$$f(z,\bar{z}) = \phi(x,y) + i\psi(x,y); \quad \bar{f}(z,\bar{z}) = \phi(x,y) - i\psi(x,y) \tag{2}$$

where $\phi(x,y)$ is the real velocity potential and $\psi(x,y)$ the stream function. In the case of a subsonic adiabatic flow past a configuration of bodies, successive approximations of the complex velocity potential are obtained by evaluating the terms of the expansion,

$$f(z,\bar{z}) = f_0(z,\bar{z}) + M_0^2 f_1(z,\bar{z}) + M_0^4 f_2(z,\bar{z}) + \ldots \tag{3}$$

where

$$M_0 = q_0/a_0 < 1 \tag{4}$$

Here q_0 is a reference velocity and a_0 the speed of sound in the fluid at rest. The first approximation is found to be an analytic function of the variable z alone,

$$f_0(z,\bar{z}) = f_0(z) \tag{5}$$

The function of $f_0(z)$ is the complex potential of the incompressible flow that fulfills the same conditions (at the body surfaces, at large distances, and for the circulation) as the subsonic flow. The second approximation can be evaluated in terms of $f_0(z)$, and we have

$$f_1(z,\bar{z}) = \frac{1}{4q_0^2}\left[\frac{\mathrm{d}f_0}{\mathrm{d}z}\overline{\int_{z_1}^{z}\left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2 \mathrm{d}z} + g(z)\right] \tag{6}$$

where z_1 is an arbitrary point in the flowfield and $g(x)$ a generally multivalued analytic function of z. The function $g(z)$ is determined by using the boundary conditions and by requiring that the velocity is single valued in the flowfield. Higher approximations can be calculated, but they grow in complexity at a tremendous rate. In what follows, we shall consider the second approximation only.

Aerodynamic Forces

The force on a body due to the flow past it is evaluated by integrating the pressures and, in the case of blowing or suction, by adding the rate of momentum flux through the body surface. Using the complex variable notation for two-dimensional flow, we have

$$X - iY = -\oint_{\mathrm{C}} p\,\mathrm{d}\bar{z} - \rho_0 \oint_{\mathrm{C}} (u - iv)\,\mathrm{d}\psi \tag{7}$$

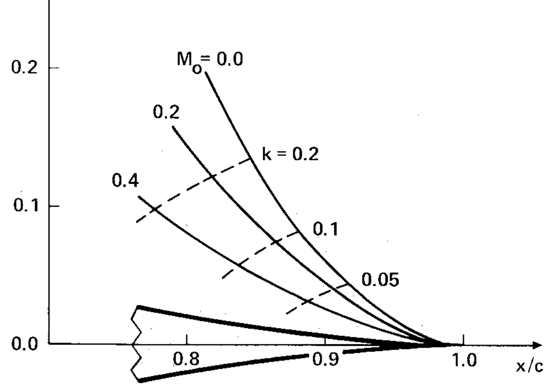

Fig. 1 Stationary vortex positions above the trailing edge of symmetric Joukowski airfoil at zero angle of attack.

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where X and Y are the components of the aerodynamic force, p the pressure, ρ_0 the stagnation density, u and v the velocity components, and $\mathcal{C}$ the contour of the body.

When the flow is calculated with the aid of the Imai-Lamla method, Eq. (7) yields the complex force in the form of a power expansion

$$X - iY = X_0 - iY_0 + M_0^2(X_1 - iY_1) + \ldots \tag{8}$$

To evaluate the terms in Eq. (8), first the velocity components and the pressure are expressed with the aid of the complex velocity potential. For the complex velocity $u - iv$, we have[3]

$$\begin{aligned} u - iv &= \frac{\mathrm{d}f_0}{\mathrm{d}z} + M_0^2\left(\frac{\partial f_1}{\partial z} + \frac{\partial \bar{f}_1}{\partial z}\right) + \mathcal{O}(M_0^4) \\ &= \frac{\mathrm{d}f_0}{\mathrm{d}z} + \frac{M_0^2}{4q_0^4}\left[\frac{\mathrm{d}^2 f_0}{\mathrm{d}z^2}\overline{\int_{z_1}^{z}\left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2 \mathrm{d}z} \right. \\ &\quad \left. + \left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2\left(\overline{\frac{\mathrm{d}f_0}{\mathrm{d}z}}\right) + \frac{\mathrm{d}g}{\mathrm{d}z}\right] + \mathcal{O}(M_0^4) \end{aligned} \tag{9}$$

The required expression for the pressure is obtained by expanding the isentropic relation between the pressure and the velocity $q = \sqrt{u^2 + v^2}$ in powers of M_0^2. This gives

$$p = p_0 - \frac{1}{2}\rho_0 q_0^2\left[\frac{q^2}{q_0^2} - \frac{1}{4}M_0^2\frac{q^4}{q_0^4} + \mathcal{O}(M_0^4)\right] \tag{10}$$

where p_0 is the stagnation pressure. Now, the aerodynamic forces are expressed in terms of the complex velocity potential by combining Eqs. (2-7), (9), and (10). This readily yields

$$X_0 - iY_0 = \frac{i\rho_0}{2}\oint_{\mathcal{C}}\left(\frac{\mathrm{d}f_0}{\mathrm{d}z}\right)^2 \mathrm{d}z \tag{11}$$

Equation (11) is the classical Blasius formula for the first incompressible approximation. The evaluation of the second term in the expansion of the aerodynamic forces involves laborious calculations. Finally, we obtain

$$X_1 - iY_1 = \frac{\rho_0}{4q_0^2}\oint_{\mathcal{C}}\frac{\mathrm{d}f_0}{\mathrm{d}z}\left[h(z) + \frac{X_0 + iY_0}{2\pi\rho_0(z - z^*)}\,\frac{\mathrm{d}f_0}{\mathrm{d}z}\right]\mathrm{d}z \tag{12}$$

Here z^* is a point chosen at will inside the body contour $\mathcal{C}$ and

$$h(z) = \frac{\mathrm{d}g}{\mathrm{d}z} + \frac{X_0 + iY_0}{\pi\rho_0}\ell n(z - z^*) \tag{13}$$

Since the complex velocity given by Eq. (9) is single valued in the flowfield, the function $h(z)$ is analytic and single valued in any flow region not including other bodies. Equation (8), with $(X_0 - iY_0)$ and $(X_1 - iY_1)$ obtained from Eqs. (11) and (12), can be regarded as Blasius theorem with first-order compressibility corrections for subsonic speeds. The resulting aerodynamic forces are expressed with the aid of integrals on the body contour of single-valued analytic functions, so that the integration path can be enlarged, provided that no other bodies are included inside. In evaluating the integrals, the theorem of residue can be used.

To conclude the evaluation of the aerodynamic forces, we note that the subsonic Blasius theorem also holds when there are compressible vortices in the flow, provided that the supersonic flow regions around the vortices do not intersect and do not reach the bodies. In this case, a uniformly valid second-order approximation of the compressible flow is obtained by matching the exact solutions for isolated vortices with an Imai-Lamla approximation.[5] Then, the force acting on a body or on a vortex can be evaluated with the aid of the subsonic Blasius theorem by choosing the integration paths in the subsonic flow region.

An Example

We applied the extension of the Blasius theorem to determine the position of a stationary spanwise vortex trapped by an airfoil in compressible flow. We considered a Joukowski airfoil and a free vortex situated in a uniform stream. For this configuration, the incompressible velocity potential $f_0(z)$ is known.[6] Starting from this incompressible result and assuming that the velocities at the airfoil are subsonic, the second-order approximaiton of the compressible velocity potential was calculated by using the Imai-Lamla method. In particular, the function $h(z)$ involved in the evaluation of the aerodynamic forces was obtained in close terms. Then, the force acting on the vortex was evaluated with the aid of the extension of Blasius theorem. For this purpose, the integration path in Eqs. (10) and (11) was taken as a closed curve going around the vortex in the subsonic flow region. The stationary position of the vortex was determined by requiring that no force acts on the vortex. From this condition, a set of two simultaneous equations for the coordinate of the vortex was obtained and the solution of this set of equations was computed numerically.

Figure 1 shows the loci of the stationary vortex positions above the trailing edge of a symmetric Joukowski airfoil set at zero angle of attack. The chord of the airfoil is c and its thickness ratio is 0.2. The nondimensional coordinates x/c and y/c of the vortex depends on the Mach number M_0 of the stream and on the nondimensional vortex strength.

$$k = \Gamma/2\pi q_0 c \tag{14}$$

where Γ is the circulation around the vortex and q_0 the velocity of the stream. The effects of compressibility on the position of the vortex are found to be significant and their trend is to move the vortex upstream and closer to the airfoil.

Conclusions

The Blasius theorem has been extended to subsonic speeds by deducing first-order compressibility corrections for the aerodynamic forces. The extended subsonic Blasius theorem retains the main features of the classical theorem, since it involves only contour integrals of single-valued analytic functions.

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